Optimal Congestion Tolling Problem under the Markovian Traffic Equilibrium

Noriko Kaneko[a], Daisuke Fukuda[b], Qian Ge[c]

[a] *ex Tokyo Institute of Technology, Japan*
[b] *Tokyo Institute of Technology, Japan*
[c] *Southwest Jiaotong University, China*

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1. Introduction

We explore the optimal congestion tolling problem under the framework of system optimal Markovian traffic assignment. The Markovian traffic equilibrium (MTE) was initially proposed by Baillon and Cominetti (2008) for the analysis of the traffic equilibrium assuming that drivers make route decisions on the basis of sequential choices of links leading to his/her destination. When the recursive logit route choice model (Fosgerau et al., 2013) is employed, the MTE is equivalent to the logit Markovian assignment developed by Akamatsu (1996). The MTE therefore belongs to a broader class of equilibrium assignment models that can incorporate different types of discrete choice model for sequential (link) choice decisions that constitute a route choice without enumerating route alternatives.

Baillon and Cominetti (2008) introduced an equivalent optimization problem to the MTE and proposed efficient solution algorithms to solve the problem. Their work is the motivation for a novel optimal congestion tolling and its computation method proposed in this study. We theoretically analyze the optimal congestion tolling and system optimization under the MTE. The system optimum property and its relevance to congestion tolling (i.e., first-best tolling) was analyzed by Yang (1999) under the logit-based stochastic user equilibrium (SUE) by formulating a network-level utility. With analogy to Yang's approach, we prove that the first-best link toll under the MTE with a logit-based link choice model is identical to marginal-cost tolling for all links, which would be the same result as in the case of the logit-based SUE proved by Yang (1999).

Variables used in the analysis are summarized in Table 1.

Email addresses: noriko_kaneko@mri.co.jp (Noriko Kaneko), fukuda@plan.cv.titech.ac.jp (Daisuke Fukuda), geqian@swjtu.edu.cn (Qian Ge)

Table 1: List of variables

$G = (N, A)$	network setting (N is the set of nodes, A is the set of links)
G_d	sub-network that can be used by the users travelling towards destination d ($G_d = (N_d \bigcup d, A_d)$
$a_{ij} \in A$	link from node i to node j
$d \in D$	destination node ($D \in N$)
β_i	dispersion parameter at node i
ε	error term
m	traveler $m \in M$
v_m	instantaneous utility of traveler m
$V_m^d(j)$	value function to destination d when traveler m is at node j
t_{ij}	travel cost from node i to node j
$\tilde{t_{ij}}$	perceived travel time cost including random variables
τ_j	downstream travel cost at node j
φ_i^d	expected downstream utility of node i; i.e., $\mathbb{E}\left(\min_{a \in A_i^+}\left[z_a^d + \varepsilon_a^d\right]\right)$
z_{ij}^d	total cost from node i to destination d ($t_{ij} + \tau_j$)
P_{ij}^d	probability that a traveler at node i chooses node j
$a \in A$	state link
$k \in A$	action link
x_{ij}^d	link traffic volume whose destination is d on link a (node $i \rightarrow$ node j)
n_i^d	node traffic volume whose destination is d on node i
g_i^d	demand vector originating in node i towards d
w_{ij}	aggregated link traffic volume for each destination $d \in D$
s_{ij}	link cost function
k	number of iterations of the fixed point algorithm
α	step size of the fixed point algorithm
A_i^+	outgoing links at node i
A_i^-	entering links at node i
U	traveler's benefit
TC	total cost
MC	marginal cost
C_a	capacity of link a
t^0	free travel time
λ_i	Lagrange multiplier at node i
δ_{in}	dummy variable when the node is the initial node
δ_{tem}	dummy variable when the node is the terminal node
$\bar{t}_a\left(v_a\right)$	$t_a\left(v_a\right) + \hat{t}_a\left(v_a\right)$, where $\hat{t}_a\left(v_a\right) = v_a \cdot \frac{\partial t_a(v_a)}{\partial v_a}$

2. MTE

According to Baillon and Cominetti (2008), the MTE is defined as the solution to the following fixed point problem (Eqs. (1)) with respect to the variables $\left(w_{ij}, t_{ij}, z_i^d, x_{ij}^d, n_i^d\right)$.

$$MTE : \begin{cases} t_{ij} = s_{ij}(w_{ij}) & \forall (i,j) \in A \\ z_i^d = t_{ij} + \tau_j^d & \forall i \in N_d,\ d \in D \\ \tau_i^d = \varphi_i^d(t_{ij} + \tau_j^d) & \forall i \in N_d,\ d \in D \\ n_i^d = g_i^d + \sum_{k \in N_d^-(i)} x_{ki}^d & \forall i \in N_d,\ d \in D \\ x_{ij}^d = n_i^d P_{ij}^d & \forall (i,j) \in A_d,\ d \in D \\ w_{ij} = \sum_{d \in D, (i,j) \in A_d} x_{ij}^d & \forall (i,j) \in A \end{cases} \tag{1}$$

Further, the MTE is formulated as an equivalent minimization problem as shown in Eq. (2).

$$\min_t \phi(t) \triangleq \sum_{ij \in A} \int_0^{t_{ij}} s_{ij}^{-1}(z)dz - \sum_{d \in D} g_i^d \tau_i^d(t) \tag{2}$$

It is possible to set up the dual problem as expressed in Eq. (3).

$$\begin{aligned} &\min_{(w,x) \in V} \sum_{i,j \in A} \int_0^{w_{ij}} s_{ij}(z)dz - \sum_{d \in D} \chi^d \left(x^d\right) \\ &\text{where } \chi^d \left(x^d\right) = -\sup_{z^d} \sum_{ij \in A} \left(\varphi_i^d \left(z^d\right) - z_j^d\right) x_{ij}^d \end{aligned} \tag{3}$$

3. System Optimum Congestion Tolling

We formulate the optimal congestion tolling problem as a system optimum problem under the Markovian assumption of drivers' route choice behavior. More specifically, the link choices of drivers at each node i towards destination d are determined by a multinomial logit model with a node specific dispersion parameter θ_i^d with the consideration of tolling for all links in the network. This fundamental idea is in line with Yang (1999), who analyzed system optimum congestion tolling under a path-based logit SUE. Likewise, we assume that the net economic benefit for all drivers under the MTE can be measured as the traveler's total gross benefit minus the total cost (i.e., $U - TC$). The Markovian system optimal problem is then formulated as the maximization problem of the objective function $(U - TC)$ with respect to the flow conservation at each node as follows.

Markovian System Optimal

$$
\begin{aligned}
\text{maximize } &: Z(x) = -\sum_{d\in D}\sum_{i\neq d}\frac{1}{\theta_i^d}\left[\sum_{ij\in A_i^+} x_{ij}^d \ln\left(x_{ij}^d\right) - \left(\sum_{ij\in A_i^+} x_{ij}^d\right)\ln\left(\sum_{ij\in A_i^+} x_{ij}^d\right)\right] \\
&\qquad - \sum_{i,j\in A} t_{ij}\left(w_{ij}\right)\cdot w_{ij} \\
\text{subject to } &: \quad g_i^d + \sum_{k\in N_d^-(i)} x_{ki} = \sum_{ij\in A_i^+} x_{ij} \\
&\qquad x_{ij}^d \geq 0 \\
&\qquad \forall d \in D \\
&\qquad \forall (i,j) \in A
\end{aligned} \tag{4}
$$

The objective function of the above-mentioned problem is strictly convex and the constraints form a non-empty convex set. There therefore exists a unique optimal solution in the feasible set. The Karush–Kuhn–Tucker conditions for any x_{ij} are thus sufficient to obtain the solution. The Lagrangian of the Markovian system optimal problem reads as follows.

$$
\begin{aligned}
&\frac{1}{\theta_i^d}\left(1+\ln x_{ij}\right) - \frac{1}{\theta_i^d}\left(1+\ln\left(\sum_{j'\in A_i^+} x_{ij'}\right)\right) + T_{ij} + \sum_{ij\in A}\left(\lambda_i\delta_{term} - \lambda_j\delta_{in}\right) = 0 \\
&\qquad d\in D, \quad (i,j)\in A
\end{aligned} \tag{5}
$$

The number of Lagrange multipliers is equal to number of nodes because there are constraint conditions at each node. We additionally define the variable T_{ij} according to Eq. (6).

$$
\begin{aligned}
T_{ij} &:= \sum_{(i,j) in A} \overline{t_{ij}}(w_{ij})\delta_{ij,d} \\
&= \frac{\partial \sum_{ij\in A} t_{ij}\left(x_{ij}\right) x_{ij}}{\partial x_{ij}} \\
&= t_{ij}\left(x_{ij}\right) + \frac{\partial t_{ij}\left(x_{ij}\right)}{\partial x_{ij}} x_{ij} \\
&= t_{ij}\left(x_{ij}\right) + MC_{ij}
\end{aligned} \tag{6}
$$

To transform Eq. (5), the link choice probability is finally reformulated as Eq. (7).

$$
\begin{aligned}
P_{ij}^d &= \frac{x_{ij}}{\sum_{ij\in A} x_{ij}} \\
&= \frac{\exp\left(-\theta_i^d\left(T_{ij} - \frac{1}{\theta_i^d}\ln\left(\sum_{j'\in A_i^+} e^{-\theta_i^d z_{j'}^{\prime d}}\right)\right)\right)}{\sum_{ij\in A}\exp\left(-\theta_i^d\left(T_{ij} - \frac{1}{\theta_i^d}\ln\left(\sum_{j'\in A_i^+} e^{-\theta_i^d z_{j'}^{\prime d})}\right)\right)\right)} \\
&= \frac{\exp\Big(-\theta_i^d\left(t_{x_{ij}} + MC_{ij} + \tau_j'\right)\Big)}{\sum_{ij\in A}\exp\Big(-\theta_i^d\left(t_{x_{ij}} + MC_{ij} + \tau_j'\right)\Big)}, \quad d \in D
\end{aligned} \tag{7}
$$

Equation (7) is similar to the link choice probability in the MTE. This indicates that, from the viewpoint of maximizing social welfare, user externality exists in the system optimality conditions. Therefore, by imposing a marginal-cost toll at each link exactly equal to the externalities, we can ensure that the users' optimal private choices are also socially optimal in terms of the maximization of net economic benefit. Consequently, the classical principle of traditional marginal-cost pricing is still applicable in the MTE situation. This conclusion coincides with the findings by Yang (1999) for optimal link tolls under a logit-based SUE.

4. Numerical Examples

We demonstrate the properties of our theoretical derivation using the Nguyen–Dupuis network and the Chicago Sketch network. For brevity, only the former result is discussed. Unlike the method of successive average used in Baillon and Cominetti (2008), we propose a novel computational method based on the value iteration approach by referring to Mai et al. (2015) to numerically obtain the fixed point with or without optimal congestion tolling.

Link cost functions are given by the standard function of the Bureau of Public Roads written as Eq. (8).

$$
t_a = t_a^0\left[1 + 0.15\left(\frac{x_a}{C_a}\right)^4\right], \ a = 1, 2, ...19 \tag{8}
$$

Table 2 and Figure 1 present the assignment results. We note that the dispersion parameters are set in inverse proportion to the link length, and the range is set as $0.5 \leq \theta_i^d \leq 1.0, \forall i, \ d$.

Table 2: Comparison of MTE-UE/MTE-SO (Nguyen–Dupuis network)

	MTE-UE	MTE-SO	MTE-UE/MTE-SO
Total travel time (min.)	78,112	72,694	1.07
Traveler's benefit (min.)	2,244	2,538	0.88
Social benefit (min.)	-75,868	-70,156	1.08
Total revenue (min.)	0	82,450	-

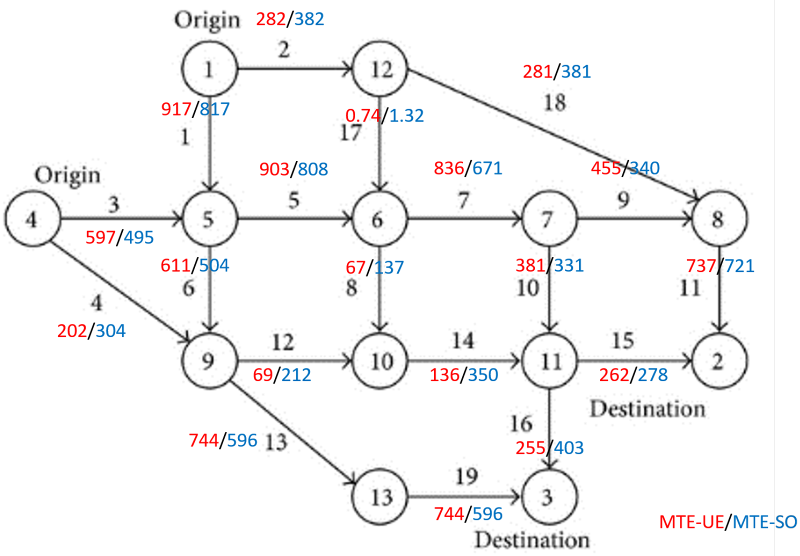

Figure 1: Assignment result of MTE-UE/MTE-SO (Nguyen–Dupuis network)

Comparing MTE-UE and MTE-SO, the improvement in social benefit achieved by imposing marginal cost is confirmed. The traveler's route choice has more variance in MTE-SO. As a result, the traffic congestion may be alleviated and the total travel time could be reduced, leading to an increase in social welfare.

5. Conclusions and Future Works

This study proved that the classical principle of marginal-cost pricing still holds under the Markovian route choice framework from the viewpoint of maximizing social benefit. We further numerically demonstrated the property of the theoretical derivation with the novel and efficient computational approach. The proposed congestion tolling scheme, however, is is formulated in the framework of the static traffic assignment which is insufficient to alleviate the propagation of traffic congestion. Such extension to the dynamic setting is important future work.

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