Mobility Service Providers’ Interacting Strategies under Multi-modal Equilibrium

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In this paper, we analyse the interactions between Mobility Service Providers (MSP) strategies in a multi-modal system. We formulate the problem using a bi-level structure extending Multi-modal Network Design (MND) principles. The upper level models the profit maximisation objectives of multiple MSPs operating and offering different services to the users. At the lower level, the users are divided into heterogeneous classes based on their travel characteristics, and assigned to a multi-modal super-network characterised by non-separable cost functions to account for the interaction between modes. The problem is formulated as an Equilibrium Problem with Equilibrium Constraints (EPEC), which is characterised by a Variational Inequality (VI) Problem at the lower level, and a novel Nash equilibrium condition is proposed for the upper level. We develop an iterative solution algorithm that combines the classical Diagonalization method with an adaptive Extragradient Method to solve this complex problem and test it on simple examples, which demonstrate the validity of the approach in identifying the range of solutions where MSPs can have potentially profitable businesses, and quantify the Price of Anarchy due to MSPs competition. The results show also the existence of several equilibria for the same problem, strongly influenced by variations in pricing schemes, demand definition or MSP strategic decisions.

1 Introduction

In recent years, new mobility solutions, including car-sharing, bike-sharing, carpooling, etc. are promoted as alternatives to private cars or as a means for first/last mile support to public transport (PT) (Cohen and Shaheen, 2018; Shaheen and Chan, 2016). However, providing users with such a wider range of multi-modal transportation choices has increased competition between mobility service providers (MSPs) (Garus

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et al., 2022). In this context, concepts like Mobility-as-a-Service (MaaS) have emerged as a ways to seamlessly integrate multiple services that can be accessed and paid for through a single subscription (Heikkilä, 2014), with the purpose of offering a portfolio of mobility solutions to meet travellers’ needs, and in turn facilitating and encouraging the use of public transportation and shared mobility options as alternative to privately owned vehicles. However, successful implementation of such solutions must overcome several challenges, including the complexity of integrating all services, define effective revenue sharing agreements and schemes, which require collaboration among multiple service providers across different sectors that are often in competition (Butler et al., 2021), and will depend on a range of factors, including user convenience and affordability, and the availability and reliability of transportation options (Hoerler et al., 2020). To properly assess these solutions, it is crucial to develop models that take into account factors like users’ needs and constraints, the availability and composition of different mobility services, and the strategies that can be adopted by different MSPs to establish a profitable business. Such models would provide a framework for evaluating the real-world applicability of MaaS and determining its impact on various actors, including users, public transport operators, and private mobility service providers.

The contribution of this paper is a novel mathematical approach with a bi-level structure able to capture the interaction between multiple MSPs, and users with various characteristics and needs. At the lower level, we formulate a multi-modal equilibrium model that includes mode-specific congestion effects and captures interdependent link costs arising from the presence of different transportation modes and user classes, which represent travellers’ heterogeneity in terms of mode choices and travel preferences (Pantelidis et al., 2020). To capture mode choice constraints due to activity and trip chaining strategies, the users are assigned to the multi-modal network using a super-network structure. In this approach, link cost functions are assumed non-separable to account for the interaction between modes, consequently the equilibrium condition is formulated as a Variational Inequality (VI) problem. At the upper level we consider multiple MSPs, whose objective functions depend on the lower level flows. MSPs may cooperate or compete to attract a share of the market and maximise their own profit. The upper level is also formulated as an equilibrium problem, for which the lower level equilibrium acts as a constraint. The chosen representation allows the concept of mobility packages to be directly encoded within multi-modal network paths that enables, for example, representation of cooperation and revenue sharing among different MSPs. Given the interaction between users at the lower level, and between MSPs at the upper level, the structure of the problem takes the form of an Equilibrium Problem with Equilibrium Constraints (EPEC), where at the upper level a Nash game is established between the different MSPs, subject to Wardrop equilibrium at the lower level.

To the best of the authors’ knowledge, this marks the first application of EPEC within a multi-modal context, wherein MSPs offering different mobility services may engage in competition or collaboration. Generally, the proposed methodology allows to evaluate MSPs’ strategies and users’ choices to predict the macro economic impact in the transportation market. Hence, the model developed in this paper is intended to be on a strategic, long-term planning horizon, rather than for operational management applications. We showcase the potential of the model through simple examples, which allow us to study MSPs interacting strategies, investigate the implications of new market entrants, and to assess the impact of increased overall travel demand, among other application opportunities. Thanks to the capability of including and incorporating complex interactions between actors and modes of transport, and to model the relation between MSPs decision-making and resource allocation processes, this model can provide insights into market strategies that can be employed by MSPs to ensure profitability while maintaining affordability for the users. Moreover, the modelling approach allows investigating different cooperation and competition settings and services integration, offering a valuable tool for economic assessment and for policy making.

The rest of the paper is structured as follows. Section 2 introduces the background motivating our
methodological approach, with a literature review on MaaS and on existing approaches to model multi-actor systems in transportation. Section 3 defines the model assumptions, and the MSPs’ and users’ formulations employed in this study, and the EPEC formulation for the bi-level system, with its solution algorithm applied to two different examples in Section 4. Finally, Section 5 discusses the conclusions drawn from the study and potential future developments of the methodology.

2 Background

MaaS-bundled multi-modal services offer users a range of mobility services through a single mobile application. These services are sold at different prices depending on the combination of transportation modes included in the packages. Although this concept has been extensively discussed and tested in different pilots, few studies have attempted to model and assess its impact on transportation networks, a key reason being that it is a complex system involving various actors with diverse goals who coexist and interact in the same ecosystem and whose choices influence each other (Kamargianni and Matyas, 2017). However, one of the primary barriers in implementing MaaS is getting the various MSPs to cooperate (Meurs et al., 2020). Regulators have the fundamental role of creating policy frameworks for equitable competition and cooperation, service accessibility and quality (Kamargianni and Matyas, 2017), through incentives such as subsidies or parking regulations (Karlsson et al., 2020). In this context, the business models implemented by different MSPs must be modified to maintain a profitable service (Polydoropoulou et al., 2020) and to offer packages that reflect the users’ daily travel needs and their willingness to pay (Ho et al., 2018).

The dynamics and interactions between various actors make the MaaS system challenging to model. In addition, MaaS is not solely a multi-actor system; it also involves multiple mode choices on the users’ side and various forms of implementation of mobility bundles, which result in partial cooperation between MSPs. To overcome some of these issues, in the literature the MaaS ecosystem has been studied as a two-sided market, in which a digital platform is seen as a means of enabling interactions between two parties and must be attractive to both MSPs and users (Meurs and Timmermans, 2017). Extending their one-side simulation-based method (Djavadian and Chow, 2017b), Djavadian and Chow (2017a) proposed an agent-based stochastic user equilibrium (SUE) under a two-sided flexible transport market. Flexible transport services (FTS) (such as demand-responsive services, shared services and taxis) are modelled as sellers, users are buyers and the platform represents the built environment. This simulation process tends to adjust users’ choices and FTS’s operating policy. However, in the application the model does not incorporate all the diverse mobility services available in the area, and it does not include package subscriptions. Xi et al. (2022) modelled the two-sided MaaS market as a single-leader multi-follower game. Specifically, the MaaS regulator (platform) is considered to be the leader with MSPs and travellers as two groups of followers. The leader aims to maximise profits by adjusting prices for travellers and MSPs, and creating MaaS bundles. Travellers seek to minimise their travel costs by selecting the most convenient means of transport within the MaaS system, whereas MSPs aim to maximise profits by choosing the proportion of their mobility resources supplied to MaaS. The number of participating MSPs and traveller requests depends on the prices set by the leader and other participants. The methodology uses a name-your-own-price auction-based mechanism, where users set their bid to have access to MaaS, and the regulator accepts or rejects them based on internal threshold price. While this approach effectively represents the dynamics of the platform, it does not account for potential interactions with modes of transportation that are not integrated into the platform. Pantelidis et al. (2020), characterised the MaaS problem as a many-to-many assignment game. In their model, users are paired with a viable path that connects their origin and destination (OD). These paths can potentially involve multiple MSPs, each of whom possess one or more links within the network. Through numerical
examples, different scenarios are analysed, such as the entry of a new competitor in the market or the acquisition of one of the two operators from a government agency. However, the problem is formulated as a matching problem in which congestion effects are ignored. More recently, van den Berg et al. (2022) developed an economic framework to analyse the effect on pricing, profits and welfare based on different MaaS strategies. Specifically, they defined a model in which two mobility services are represented through a supply chain structure with four links connecting an OD. In this setting, they could analyse what happens with and without MaaS platform adopting different strategies. Nevertheless, congestion is not taken into account as well as the heterogeneity of users and their choices.

Traditionally, when studying the interactions between operators and travellers in transportation research, the focus has been on single-mode networks within the context of network design modelling (Fara-hani et al., 2013). However, in order to represent more realistic scenarios, there has been a shift towards incorporating multi-modal networks into these studies. For example, combining automobile and public transport networks in relation to Park and Ride (P+R) facilities (García and Marin, 2002; Fan et al., 2014; Ye et al., 2021), focusing on assessing the optimal placement of pedestrian sidewalks or streets within automobile and transit systems (Wu et al., 2005; Rashidi et al., 2016), or establishing relationships between public transportation and vehicle sharing systems in the context of first/last mile connection (Nair and Miller-Hooks, 2014; Pinto et al., 2020). However, these works include only a limited selection of modes of transport. More recently, Najmi et al. (2023) have developed a model able to include six modes of transport: private vehicle, walking, ride-sourcing, ridesharing-as-driver, ridesharing-as-rider, and public transport. Their strategic model simultaneously considers ride-sourcing trying to maximize their profit, while a network operator controls congestion distribution, and multi-class users are assigned to the network.

Currently, there is limited research that includes multiple MSPs, coexisting, competing or cooperating. From an economic and strategic point of view (i.e. to broadly understand if a business has potential profitability and market value), it is essential to model the interactions between MSPs and travellers of the transport network to predict the response of these actors as a consequence of the variation in strategies of the entire system. In particular, scenarios offering a new transport service, introducing new regulations/incentives, or increasing users’ heterogeneity, could substantially change the network equilibrium. Some works have analysed this type of problem. Zhou et al. (2005) investigate transit competition with a bi-level equilibrium formulation presented as a Stackelberg game, considering elastic demand assigned under Stochastic User Equilibrium (SUE). In the context of fast charging stations for electric vehicles, Guo et al. (2016) developed a Multi-agent Optimization Problem with Equilibrium Constraints (MOPEC)-based model to study interactions between multiple competitive investors and travellers assigned to a congested transport network. Jiang et al. (2020) propose a game-theoretical model to study market competition between two bike-sharing companies. Yang et al. (2022), instead, defined a bi-level model to optimise pricing and relocation in a competitive one-way car-sharing market.

The significance of properly addressing the competition and cooperation strategies among firms or MSPs in the transportation market is evident, but we argue that there is a lack of studies that examine the different forms of competition and collaboration emerging in a multi-modal system, or arising from the co-existence of multiple service providers in the same market. Furthermore, we observe that existing studies predominantly concentrate on uni-modal networks with homogeneous demand, neglecting the cost/congestion interaction between different modes.

The modelling approach proposed in this paper aims to address the above research gaps. Starting with the complex structure of multi-modal systems, our approach aims to capture possible strategies adopted by MSPs when competing in the transportation market, or cooperating through mobility packages. These strategies are influenced by the modal choices made by heterogeneous users within a congested multi-modal
network. We seek to identify and understand the equilibrium solutions that occur, in which no MSP and no user want to change their strategies. To fill the above gaps, this paper presents a novel modelling approach to understand the impact of the strategies employed by multiple MSPs that offer diverse mobility services. The study takes into account the potential aggregation of these services into mobility packages, targeting heterogeneous groups of users, minimising their travel costs while performing their daily trip chains. In our model each MSP influences other players’ strategies of seeking to maximise their individual profits, and determine the transportation offer proposed to users. Meanwhile, multiple classes of users interact through congestion effects, and make travel choices minimising their own travelling costs.

3 Methodology

3.1 Model Structure and Assumptions

In this section, a multi-modal network model is introduced with the structure of an expanded network (super-network), where users’ daily trip chains are explicitly modelled and the different mobility services available in the area are included in a multi-layered graph structure. Since our focus is on long-term impacts, e.g. in terms of profitability of a certain service and how it can be optimised, the model developed is static, representing a typical weekday, and spatially aggregated at a zonal (or regional) scale. Table 1 provides the model notation.

Let travellers be divided into $K$ classes based on their personal attributes and daily trip chains (e.g. leave home to drop off their children at school, go to work, pick up their children for shopping, and finally return home). Users of the same class follow the same sequence of visited zones but may differ in how they travel between these locations. It is then possible to define a trip-based network where the sequence of trips between zones is modelled as a directed graph. In Figure 1 an example of this network is illustrated considering one class of users performing a certain sequence of activities $k$. Here, the different visited locations are explicitly identified with the activities performed at each zone for illustration purposes. However, users may perform different activities; the user class is mainly determined by the sequence of visited locations.

![Figure 1: User Class $k$ with daily trip chain](image)

Let a node $n$ correspond to a location (zone). A link $a$ can be a modal link representing a trip from one location to another, or can be a subscription link. We assume that modal link is managed by a single MSP, with $A_j^l$ the vector of links owned by MSP $j$. Subscription links can be shared between different MSPs as part of the same mobility package. The vector of subscription links involving $j$ is $A_j^s$.

There have been numerous studies on the construction of super-networks for modelling daily activity-travel patterns (Lo et al., 2003; Feixiong Liao and Timmermans, 2011; Vo et al., 2020), and applied to assignment and design problems in large-scale multi-modal transportation networks. However, in our approach, we do not seek to construct super-networks that explicitly model all mode-specific alternative routes of an individual in the multi-modal system since our focus is on mode choice rather than route choice.
Table 1: Model Notation

<table>
<thead>
<tr>
<th>Sets</th>
<th>Indices</th>
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</thead>
<tbody>
<tr>
<td>( J ) Set of MSPs</td>
<td>( j ) a MSP</td>
</tr>
<tr>
<td>( K ) Set of user classes</td>
<td>( k ) a user class</td>
</tr>
<tr>
<td>( S ) Set of mobility subscriptions</td>
<td>( s ) a mobility subscription</td>
</tr>
<tr>
<td>( A ) Set of links</td>
<td>( a ) a link of the network</td>
</tr>
<tr>
<td>( A^j ) Subset of modal-links owned by MSP ( j )</td>
<td>( n ) a location (node)</td>
</tr>
<tr>
<td>( A_s ) Subset of subscription links</td>
<td>( w ) an OD pair</td>
</tr>
<tr>
<td>( A^j_s ) Subset of subscription links involving MSP ( j )</td>
<td>( p ) a path connecting an OD</td>
</tr>
<tr>
<td>( N ) Set of locations (nodes)</td>
<td></td>
</tr>
<tr>
<td>( W ) Set of OD pairs</td>
<td></td>
</tr>
<tr>
<td>( P ) Set of paths in the network</td>
<td></td>
</tr>
<tr>
<td>( P_w ) Subset of paths between ( w \in W )</td>
<td></td>
</tr>
<tr>
<td>( X ) Set of all feasible path flows</td>
<td></td>
</tr>
<tr>
<td>( D ) Total travel demand of the network</td>
<td></td>
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<thead>
<tr>
<th>Users’ variables</th>
<th>Supplier’s variables</th>
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<tbody>
<tr>
<td>( f ) vector of link flows</td>
<td>( v ) vector of fleet size</td>
</tr>
<tr>
<td>( f_a ) flow on link ( a )</td>
<td>( v^j ) fleet size for MSP ( j )</td>
</tr>
<tr>
<td>( f_k^a ) flow of class ( k ) on link ( a )</td>
<td>( v_a ) number of vehicles on link ( a \in A^j )</td>
</tr>
<tr>
<td>( x ) vector of path flows</td>
<td></td>
</tr>
<tr>
<td>( x_p ) flow on path ( p )</td>
<td></td>
</tr>
<tr>
<td>( x_p^k ) flow of class ( k ) on path ( p )</td>
<td></td>
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</table>

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d_k^w ) demand of class ( k ) on OD ( w )</td>
<td>( c^j_{\text{lease}} (\sum_{a \in A^j} v_a) ) leasing cost (€) for MSP ( j )</td>
</tr>
<tr>
<td>( \gamma^j ) relocation factor for MSP ( j )</td>
<td>( t_{a,\text{access}} (f_a, v_a) ) access time (hour/day)</td>
</tr>
<tr>
<td>( l_a ) length of link ( a ) (km)</td>
<td>( t_{a,\text{main}} (f) ) time in the main mode of transport (hour/day)</td>
</tr>
<tr>
<td>( c_{a,s} ) daily cost for subscription ( s ) (€/day)</td>
<td>( t_{a,\text{egress}} (f_a, v_a) ) egress time (hour/day)</td>
</tr>
<tr>
<td>( r_{a,s} ) daily subsidy based subscription ( s ) (€/day)</td>
<td>( t_{a,\text{wait}} (f_a, v_a) ) waiting time (hour/day)</td>
</tr>
<tr>
<td>( c_{a,h} ) cost per hour ( h ) travelling on link ( a ) (€/hour)</td>
<td>( t_{a,\text{park}} (f_a, v_a) ) parking time (hour/day)</td>
</tr>
<tr>
<td>( c_{a,km} ) Cost per kilometre km on link ( a ) (€/km)</td>
<td>( C^k_a (f, v_a) ) total cost on link ( a ) for class ( k )</td>
</tr>
<tr>
<td>( c_{a,\text{fixed}} ) ticket cost on link ( a ) (€/day)</td>
<td>( C^k_{a,\text{access}} (f_a, v_a) ) total access cost on link ( a ) for class ( k )</td>
</tr>
<tr>
<td>( c_{a,\text{fuel}} ) fuel/recharge cost for vehicle (€/km)</td>
<td>( C^k_{a,\text{main}} (f, v_a) ) total travel cost on link ( a ) for class ( k )</td>
</tr>
<tr>
<td>( c_{a,\text{park}} ) cost to find a parking slot (€/hour)</td>
<td>( C^k_{a,\text{egress}} (f_a, v_a) ) total egress cost on link ( a ) for class ( k )</td>
</tr>
<tr>
<td>( \delta_{a,p} ) incidence matrix link-path</td>
<td>( C^k_p (x, v) ) total travel cost on path ( p ) for class ( k )</td>
</tr>
<tr>
<td>( \delta_{a,s} ) incidence matrix link-subscription</td>
<td></td>
</tr>
<tr>
<td>( \Xi ) link-link congestion incidence matrix</td>
<td></td>
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</table>
Hence, the trip chain graph in our modelling approach aggregates the underlying physical transportation infrastructure, reducing the problem complexity. As illustrated in Figure 2a, even for one mode the physical infrastructure may have multiple distinct routes, whereas our model represents only a single mode-specific link between two zones. Assuming that the underlying network achieves equilibrium at route level provides some justification for a representation using a single aggregated link (Connors and Watling, 2014).

The basic trip-chain network (like the one exemplified in Figure 1) is expanded into multiple uni-modal MSP-specific layers, as illustrated in Figure 2b. In this way we capture multiple modes connecting two pairs of zones/regions and/or MSPs operating each trip connection. This representation thereby takes into account all the possible modes of transport active in an area, including users’ private vehicles, e.g. car, bike, scooter. Combining users’ trip chains with MSPs’ information, the network in Figure 1 becomes a multi-modal trip chain-based super-network as shown in Figure 3, which depicts the trip-based network for user class $k$ (in black), and below it, its expansion into parallel uni-modal layers. Each layer corresponds to a specific mode of transportation. An MSP may possess one or several mobility services, and these services can be represented in the network as separate layers. The arrangement of layers and links will be influenced by factors such as whether the service has different prices, different specifications, or if it is part of a mobility package that includes one or more other services.

In Figure 3, four modes of transport are considered for illustrative purposes: car-sharing (red layer), private car (PC) (yellow layer), train (green layers), and bus (blue layer). Each traveller must choose a path $p$ through the multi-modal network in order to perform their daily trip sequence. In this example, users have several options for their daily trip chain:

- they can exclusively use the car-sharing service (red path);
- they can rely solely on their PCs (yellow path); alternatively they can park their vehicle at a P+R facility located in the school zone, use the train during the day, and pick up their cars when returning to the school location (yellow and green path);
- they may opt for a package that provides access to both bus and train services. Since the train does not cover all locations, users can either use the bus alone (blue path) or combine both modes of transportation (blue and green path).
The example illustrated in Figure 3 shows a subset of all feasible sequences of modes. For instance, vertical links could also be connecting different layers (e.g. car-sharing being used in combination with train would imply connecting these two layers with additional vertical links in the interchange locations).

Undoubtedly, the choice of dividing users into classes increases the complexity of the model. However, the socioeconomic characteristics only affect users’ perceived costs and not the network expansion. On the other hand, defining user classes based on the combinations of daily trip chains and activity locations could become a non-trivial problem in large-scale networks. If we assume that trip chains increase with the number of zones \(N\), there would be a combinatorial increase in the number of potential daily trip chains, and an exponential network expansion \((\sum_{n=1}^{N-1} \prod_{\tau=1}^{n-1} n(n-\tau))\). However, travel behaviour literature shows that during typical weekdays the majority of travellers tend to perform home-work-home tours when using PT or add an additional activity before/after work when travelling with private vehicles (Sprumont et al., 2022; Axhausen et al., 2002). Moreover, the combination of trip chains, activity sequences and locations are spatially limited and rather repetitive (Susilo and Axhausen, 2014). Therefore, focusing on the most common tours, it is possible to cover most of the travel demand of an area without introducing too much combinatorial complexity in the super-network. However, other factors that affect network scalability (i.e. the number of parallel layers) are the number of modes, mobility subscriptions, MSP participating in each MaaS package, and user classes. An increase in the number of MSPs increases the number of transportation modes modelled, which also increases the number of mobility subscriptions available. MSPs may choose either to sell their services individually, or to collaborate via packages that combine multiple modes of transportation. With \(J\) MSPs, when each \(j\) can only offer one mobility service, the number of layers is proportional to the number of MSPs. When each \(j\) offers a multiple packages, the number of layers increases exponentially as the number of packages \(s_j\) provided by each of the \(J\) suppliers increases.

The set of feasible modes connecting each OD are explicitly enumerated in this study, to ensure only alternatives that comply with the complex multi-modal constraints of the network are included (e.g. users do not have to pay two times for a package subscription or pay and use another mode of transport). A path (representing a trip chain for the specific user class \(k\)) can comprise three different types of links: 1) Access
links (black dashed links) allow users to access a mode of transport from their origin (Home on the left side), and egress from a mode of transport to reach their final destination (Home on the right side). Another important role of the access links is that they can represent a subscription package needed for accessing a specific mode of transport (e.g., a PT subscription) or the combination of integrated MSPs’ services (a MaaS bundle subscription). 2) Mode-specific links (horizontal links) indicate trips made from one activity location to another using a specific mode of transport (designated by colour). 3) Interchange links (vertical black links) instead allow users to change to another transport mode when departing from an activity location. Each (within-layer) mode-specific link includes in turn three stages of a trip. In the first stage, the users leave the activity location and walk to reach the main mode of transport. Subsequently, travellers use that mode of transport to reach their next destination. Finally, the user leaves that transport mode and walks to arrive at their next location (Arentze and Molin, 2013). Once they have performed all their trips, users return home completing their daily trip chain.

3.2 Mobility Service Providers

MSPs operate the transport and mobility services connecting the locations where users perform their activities, and seek to maximise their profits as main objective. Each MSP, denoted as $j \in J$, can manage one or more layers of the super-network, through which they collect revenues based on how many travellers use their service, while facing costs that depend on the service capacity provided e.g. number of vehicles. Hence, each MSP owns a fleet of vehicles $v^j$, which they strategically distribute across the links, $A^j$, of their network layers. The number of vehicles assigned to a specific link ($v_a$) will be considered available only for that trip connection: $v^j = \sum_{a \in A^j} v_a$. Daily (re-)location of vehicles, $\gamma^j$, is also included in the costs.

The defined revenue and cost components for a generic MSP $j$ ensure that the formulation can be applied to different mobility services such as bike sharing, PT, carpooling, car-sharing, taxi and train. Depending on the type of service, not all cost components will be needed, as shown in Table 2. MSP’s profit ($Z$) is defined as the difference between total revenues (TR - comprising fixed plus variable revenues) and total costs (TC - including fixed plus variable costs):

$$Z = TR - TC$$

**Revenues.** Fixed revenues ($FR^j$) for MSP $j$ arise from subscriptions and subsidies. The MSP receives revenues from the subscription fee paid by the users to access the service $c_{a,s}$ and from any subsidy $r_{a,s}$. Hence subscription-link flow terms appear in Equation 2:

$$FR^j(f) = \sum_{a \in A^j_i} (c_{a,s} + r_{a,s}) f_a$$

Variable revenues ($VR^j$) depend on travellers’ usage of the service:

$$VR^j(f) = \sum_{a \in A^j} (c_{a,h} f_{a,main}(f) + c_{a,km} f_a + c_{a, fixed}) f_a$$

The first term represents the time spent on-board $t_{a,main}(f)$. This term may be influenced by the flow on other super-network links that use the same physical infrastructure. These non-separable interactions between flows on different super-network links can be recorded by an incidence matrix, $\Xi$. Typically (as in the examples of Section 4) interactions will be symmetric and additive, corresponding to $\Xi$ having 0/1
entries e.g. private car travel time on a link is a function of the flow of private cars plus car-sharing cars using the same trip connection. Weighted combinations of non-separable flows could also be accommodated e.g. normalising flows to passenger car units. The link travel time, \( t_{a\text{main}} \), is then monotonic function (e.g. a BPR-type, see Equation (22)) of the \( a \)-th element of the aggregated flow vector \( \Xi_f \).\(^1\)

Note that not all modes of transport have interdependent congestion impacts, e.g. trains typically have a separate infrastructure or buses can have dedicated lanes. This travel time term is multiplied by the cost per hour \( c_{a,h} \) that travellers pay to use the mobility service, perceived as revenue by the MSP. The second term \( l_a \) indicates distance travelled, multiplied by revenue per kilometre \( c_{a,km} \). The third term \( c_{a,\text{fixed}} \) is the revenue from a fixed fee/ticket charged to the user each time the service is used.

**Costs.** MSP \( j \) experiences fixed and variable costs. The fixed costs (\( FC^j \)) are defined through a function that varies with the number of vehicles the supplier deploys on the network:

\[
FC^j(v^j) = c_{\text{lease}}(\sum_{a \in A^j} v_a) = c_{\text{lease}} v^j
\]

In the function \( c_{\text{lease}}(\sum_{a \in A^j} v_a) \), it is possible to include all the costs that do not change with the number of travellers served, and that the supplier has to bear in order to operate a mobility service. More specifically, these are related to investment costs, such as purchasing/leasing the fleet of vehicles, renting parking spaces, building charging stations, paying employees, and general legal and administrative costs for the company.

The variable costs (\( VC^j \)) for MSP \( j \) are associated to the daily operations, and they are defined as:

\[
VC^j(f) = \begin{cases} 
\sum_{a \in A^j} c_{a,\text{fuel}} l_a (1 + \gamma^j) & \text{if bus, train, car-pooling} \\
\sum_{a \in A^j} c_{a,\text{fuel}} l_a (1 + \gamma^j) f_a & \text{if car-sharing, bike-sharing, e-scooter, taxi}
\end{cases}
\]

The first cost, denoted as \( c_{a,\text{fuel}} \), represents the per-unit cost borne by the supplier for fuel (or electricity) when travellers use their service, directly tied to the distance traveled \( l_a \). When considering sharing services, these factors have to be multiplied by the number of travellers using that mode of transport \( f_a \). The additional cost component \( \gamma^j \) represents the relocation of vehicles or the return to a vehicle depot; note that \( \gamma^j \) can be zero for some MSPs.

Table 2 shows the connection between the costs and revenues components of the profit maximization associated with the different modes of transport. The orange box indicates that a specific factor could influence the costs or revenues of an MSP based on their marketing strategies. The green box represents a component always considered for that specific mode of transport. When, instead, a component does not influence the profit of an MSP empty boxes are shown in the table.

**Profit Maximization.** MSPs seek to maximize the profit of each mobility service offered. The revenue generated by an MSP depends on the number of travelers using their services \( f^j \) (i.e., the network link flows), while costs primarily depend on the size of their vehicle fleet \( v^j \), and how they are distributed among the network links. Given link flows \( f \) the profit for MSP \( j \) is \( Z^j(v|f) \) comprising terms defined above in Equations (2)-(5):

\[
\max_{v_a > 0} Z^j(v|f) = FR^j(f) + VR^j(f) - FC^j(v^j) - VC^j(f).
\]

\(^1\)For simplicity of notation we write the argument \( f \).
Table 2: Components of cost/revenues connected to modes of transport

<table>
<thead>
<tr>
<th>Factor</th>
<th>Bus</th>
<th>Train</th>
<th>Car-sharing one way</th>
<th>Car-sharing round trip</th>
<th>Bike-sharing one-way</th>
<th>E-scooter</th>
<th>Taxi</th>
<th>Car-pooling</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{f_a}$</td>
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<td>$r_{f_a}$</td>
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<tr>
<td>$c_{a,b,main(l)} f_a$</td>
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<td>$c_{a,km} a f_a$</td>
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<tr>
<td>$c_{a,fix} f_a$</td>
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<tr>
<td>$c_{lease (\sum_{a\in A} v_a)}$</td>
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<tr>
<td>$c_{a,fuel} a f_a$</td>
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</tr>
<tr>
<td>$c_{a,fuel} f_a f_a \gamma$</td>
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</tr>
</tbody>
</table>

$^a$ Green box: fixed cost component for that service; Pink: a potential cost component influenced by the MSP’s market strategies

3.3 Users of the Multi-modal Network

As already pointed out, inside the super-network, users’ trip chains are represented as paths $p$ connecting OD pairs. Paths encode the constraints of the multi-modal network, such as one-off subscription costs, and logical mode choice sequences. For this reason, the traffic network conditions will be expressed in a path-based approach and a priori path enumeration is taken into account. The computational expense and consequent limitations of path enumeration are well known. However, this study focuses on toy networks to illustrate the characteristics of the proposed methodology.

Generally, the problem is complicated by the presence of multiple classes and the interdependency between flows on parallel links of the super-network. Concretely, some super-network links represent copies of the same real transport link of the underlying infrastructure network e.g. travel time on car-sharing links is influenced by travellers using PC and vice versa. Consequently, the corresponding link costs are non-separable. The User Equilibrium (UE) is therefore formulated as a Variational Inequality (VI) (Dafermos, 1980). The decision variables are the path flows, represented by the vector $x \in X \subseteq \mathbb{R}^p$, with $p$ paths and $X$ the set of demand-feasible flows. With exogenously defined demand, $d^k_w$, for each class $k$ and OD pair $w$, the demand-feasibility constraint can be expressed as:

$$d^k_w = \sum_{p\in P_w} x^k_p \quad \forall w \in W \quad \forall k \in K$$  (7)

with class $k$ having path flows $x^k_p$ using the set of paths $P_w$ connecting $w$.

The relationship between link flow $f^k_a$ and path flow $x^k_p$ for class $k$ can be written in the usual way as:

$$f^k_a = \sum_{p\in P} x^k_p \delta_{ap} \quad \forall k \in K, \forall a \in A$$  (8)

The path flow vector is non-negative $x \geq 0$. The total flow on link $a$ is the sum of the link flows over all classes:

$$f_a = \sum_{k \in K} f^k_a \quad \forall a \in A$$  (9)
When making their choices, users will encounter different costs associated with their chosen links (i.e., the mode-specific travel alternatives to reach the destination where the activity is performed). Here we introduce a range of cost components that can be accommodated within this model. Simpler specifications could also be used, such as those illustrated in the examples of Section 4.

Trip costs (outlined in Table 1) may be charged daily or hourly. Daily costs are those linked with a fixed subscription or a daily ticket. Hourly costs arise from variable usage (in time or distance) and correspond to the time/distance users travel on a typical day. For ease of comparison, both total costs and MSP revenues are presented on a per-day basis in this study.

Each access link to a service may have a constant subscription cost or a fixed cost of ownership \( c_{a,s} \), if applicable to the mode of transport. Transfer links usually have zero cost when they are part of multi-modal paths; they can also serve as access links and thus have a constant subscription cost. Mode-specific links have two cost components: monetary costs charged by the MSP, and costs connected to access time, waiting time, congestion, etc. The latter costs are flow and capacity-dependent functions and may be valued differently by each class. In some cases, congestion depends on the demand flow of a specific mode of transport. In others, since each layer of the supernetwork represents the same underlying physical network, congestion can also be influenced by the flow of other modes of transport and by the flow of other classes of users crossing the same links. In particular, a class \( k \) will perceive the cost on a generic mode-specific link \( a \) of the network as follows:

\[
C^k_{a}(f, v_a) = C^k_{a, \text{access}}(f_a, v_a) + C^k_{a, \text{main}}(f) + C^k_{a, \text{egress}}(f_a, v_a) \tag{10}
\]

The first term of Equation (10) represents the access cost for class \( k \) to reach a mode of transport departing from an activity location. The access cost for user class \( k \) is:

\[
C^k_{a, \text{access}}(f_a, v_a) = c^k_{a, \text{walking}} t_{a, \text{walking}}(f_a, v_a) + c^k_{a, \text{wait}} t_{a, \text{wait}}(f_a, v_a) \tag{11}
\]

where the first component considers the time needed to reach the chosen mode of transport \( t_{a, \text{walking}}(f_a, v_a) \). This function could be considered as a constant value, derived for example from the average distance from a bus stop, or it can be influenced by the number of users choosing the same mode of transport in relation to the limited capacity of the service. In this situation the users could choose to go to the next location in order to find available vehicles, however at the cost increased travel time. All such link cost components \( (t_{a, \text{walking}}, t_{a, \text{park}}, t_{a, \text{wait}}) \) should be non-negative and monotonically increasing. In our examples we choose to adopt the BPR functional form: \( t_{a}(f_a, v_a) = t_0 \left( 1 + \alpha (f_a/v_a)^\beta \right) \). The monetary value of walking time depends on the users’ class, \( c^k_{a, \text{walking}} \). The second component in Equation (11) represents the time needed to wait for an available vehicle \( (t_{a, \text{wait}}(f_a, v_a)) \), with class-specific waiting cost \( c^k_{a, \text{wait}} \).

The second term in Equation (10), represents the total cost for user class \( k \) using the main mode of transport:

\[
C^k_{a, \text{main}}(f) = c_{a, \text{fuel}} l_a + c_{a, \text{km}} l_a + c_{a, \text{h}} t_{a, \text{main}}(f) + c_{a, \text{ticket}} + c^k_{a, \text{main}} t_{a, \text{main}}(f) \tag{12}
\]

The first term considers the cost for fuel (or electricity) \( c_{a, \text{fuel}} \) connected to the kilometres travelled \( l_a \). The other three components depend on cost \( c_{a, \text{km}} \) per kilometres travelled \( l_a \), the cost \( c_{a, \text{h}} \) connected to the time spent using the service \( t_{a, \text{main}}(f) \), and a fixed ticket cost \( c_{a, \text{ticket}} \). The last term is connected to the class dependent cost \( c^k_{a, \text{main}} \) associated to the time, \( t_{a, \text{main}}(f) \), spent in the mobility service. This includes any non-separable contributions to congestion as noted under Equation (3).
The last term of Equation (10) is the egress cost of class $k$ on link $a$:

$$C_{k,a,\text{egress}}(f_a, v_a) = C_{a,\text{park}}^k t_{a,\text{park}}(f_a, v_a) + c_{a,\text{walking}}^k t_{a,\text{walking}}(f_a, v_a)$$  \hspace{1cm} (13)

These costs are calculated considering the time spent to find an available parking space with a specific mode of transport ($t_{a,\text{park}}(f_a, v_a)$) and the monetary cost ($c_{a,\text{park}}^k$) class $k$ associated with this time. After leaving the main mode of transport, the time needed to reach the final destination has cost $c_{a,\text{walking}}^k$ multiplied by the time $t_{a,\text{walking}}(f_a, v_a)$.

Figure 4 details a section of the super-network of Figure 3, to illustrate how these costs are assigned based on the mode of transport considered.

![Figure 4: Cost details of a link from the supernetwork shown in Figure 3](image)

The supernetwork is thereby built to allow for the incorporation of diverse multi-modal services, restrictions and package subscriptions directly within its framework. These components collectively contribute to the non-additive nature of the path costs. It is necessary to generate paths so as to encode these constraints. Path generation could be automated but would require a bespoke algorithm, which is beyond the scope of the paper. In this paper the paths were generated by hand for each specific example.

It is possible to write the cost of a path connecting an OD pair $w$ for user class $k$ equal to the sum of its constituent link costs:

$$C_k^p(x, v) = \sum_{s \in A_k} \sum_{a \in A} c_{a,s} \delta_{a,s} \delta_{a,p} + \sum_{a \in A} C_{a,\text{park}}^k f_a, v_a) \delta_{a,p} \quad \forall p \in P_w, \forall w \in W$$  \hspace{1cm} (14)

**User Equilibrium.** Once the super-network is constructed and link costs defined, all users of each class are assigned to the network following Wardrop’s first equilibrium principle (Wardrop, 1952). As pointed out by Adler et al. (2021) and citations therein, Wardrop equilibria are equivalent to Nash equilibria under certain conditions.

Let $C_k^p(x, v)$ be the path cost function for a generic class $k \in K$, which depends on the capacities, $v$, supplied by MSPs. Then a vector of path flows $x^* \in X$ is a Wardrop equilibrium if and only if it is a solution of the VI problem:

$$\sum_{k \in K} \sum_{w \in W} \sum_{p \in P_w} C_k^p(x^*, v)(\xi_p - x_p) \geq 0 \quad \forall \xi_p \in X$$  \hspace{1cm} (15)
with
\[ x_p \geq 0 \quad \forall p \in P_w \]  \hspace{1cm} (16)

where (16) is the path flow non-negativity constraint. Given the vector of fleet sizes \( \mathbf{v} \), \( X^*(\mathbf{v}) \) represents the set of equilibrium solutions to (15).

Nagurney (2000) demonstrates the existence of a solution for a VI problem having the same structure as Equation (15). Existence of a solution can be established when the path cost functions \( C(\mathbf{x}) \) are a composition of continuous functions, acting on a closed and convex feasible region \( \mathbf{x} \in \mathbb{X} \). Criteria that are satisfied thanks to the network equilibrium conditions (Equation 7, 8, and 16). This reasoning can be extended to a path-based formulation, as is used here, following the approach of Watling (2006). In the simpler case when link cost functions are separable with respect to the total flow on the link, proof of solution uniqueness usually relies on strict monotonicity of the vector of link cost functions. However, as pointed out by Nagurney (2000), in the context of multi-class and multi-criteria models, even if all the components of cost satisfy strict monotonicity concerning the total link flow, this may not be sufficient. In addition the link cost functions are non-separable. Therefore we cannot prove uniqueness of solutions to (15).

**Solution Algorithm: The Extragradient Method.** In the literature, VIs are used as a powerful reformulation of a wide number of problems on different fields. For this reason, several algorithms with the purpose of efficiently solving VIs have been proposed. These algorithms achieve equilibrium iteratively through equilibration procedures.

The algorithm chosen for the proposed methodology is a relaxation of the Extragradient Method (EM) (Korpelevich, 1976), proposed by Khobotov (1987). The EM, often used to solve UE (Nagurney, 2000; Szeto and Jiang, 2014), is an extension of the classic gradient projection method, which solves VIs dividing them into quadratic programming sub-problems (Nagurney, 1998). Moreover, in this paper the improvement proposed by Panicucci et al. (2007) is incorporated, to prevent the step-size from becoming excessively small and ensure faster convergence. Moreover, each projection of is decomposed into smaller projections, one for each OD pair and each class, using the projection algorithm proposed by Michelot (1986).

In Algorithm 1 the different steps of the adaptive EM are listed, incorporating the projection algorithm presented in Algorithm 2. Parameters are \( \mu, \lambda \in (0, 1) \), and \( \bar{\theta} > 0 \). Initial feasible path flow vector is \( \mathbf{x}_0 \). The gap function \( (\text{Gap}) \), used as a convergence criterion, has the form of the following relative gap function (Chiu et al., 2011), often used in static traffic assignment models:

\[
\text{Gap} = \frac{\sum_{k \in K} \sum_{w \in W} \left[ \sum_{p \in P_w} (x_p^k * C_p^k(\mathbf{x}, \mathbf{v})) - d_w^k * C_{p,w,\text{min}}^k(\mathbf{x}, \mathbf{v}) \right]}{d_w^k * C_{p,w,\text{min}}^k(\mathbf{x}, \mathbf{v})} < \varepsilon
\]  \hspace{1cm} (17)

Specifically, the numerator represents the total gap, i.e. the deviation between the current assignment solution and the ideal shortest path. This value is then divided by the total shortest path solutions. When the travel time is close to the one of the shortest path, the numerator becomes close to zero, and therefore \( \text{Gap} \) will have a small value.
Algorithm 1 Adaptive Extragradient Method with Projection Algorithm

1: Input Lower-Level Parameters and Functions
2: Initialize $i = 0, \theta = \infty, \theta_0 = \bar{\theta}, x_0$
3: Select $MaxIter, \mu, \lambda, \varepsilon \{\text{Algorithm parameters}\}$
4: while $\theta > \varepsilon \text{ and } i < MaxIter$ do \{Stopping Criteria\}
5: \hspace{1em} Flow update:
6: \hspace{2em} Compute $y_i = x_i - \theta C(x_i, v)$ \{Vector of all ODs and classes\}
7: \hspace{2em} Compute $\bar{x}_i = \text{Projection}(y_i)$ \{using Algorithm 2\}
8: \hspace{2em} while $\theta_i > \lambda \frac{||x_i - \bar{x}_i||}{||C(x_i, v) - C(\bar{x}_i, v)||}$ do \{Reduce $\theta_i$\}
9: \hspace{3em} $\theta_i = \min\{\mu \theta_i, \lambda \frac{||x_i - \bar{x}_i||}{||C(x_i, v) - C(\bar{x}_i, v)||}\}$
10: \hspace{2em} Recompute $y_i = x_i - \theta_i C(x_i, v)$ \{update with reduced $\theta_i$\}
11: \hspace{2em} Recompute $\bar{x}_i = \text{Projection}(y_i)$ \{Algorithm 2\}
12: \hspace{1em} end while \{If $\bar{x}_i$ is viable\}
13: Compute $z_i = x_i - \theta_i C(x_i, v)$ \{Compute costs using $\bar{x}_i$\}
14: Compute $x_{i+1} = \text{Projection}(z_i)$ \{using Algorithm 2\}
15: Set $\theta_{i+1} = \min\{\theta_i, \lambda \frac{||x_i - \bar{x}_i||}{||C(x_i, v) - C(\bar{x}_i, v)||}\}$ \{Set $\theta$ for next iteration\}
16: Compute $\theta$ \{see Equation (17)\}
17: Increment $i$
18: end while

Algorithm 2 $\bar{x} = \text{Projection}(x)$

1: Input Lower Level Parameters and Functions
2: Input $x$ \{Input path flow from Algorithm 1\}
3: for $w = 1 : W$ do \{Number of ODs\}
4: \hspace{1em} for $k = 1 : K$ do \{Number of classes\}
5: \hspace{2em} $n = \text{length}(\bar{x}_w^k)$ \{Path flows for this class and OD\}
6: \hspace{3em} $\bar{x}_w^k = \bar{x}_w^k + \frac{[d_w^k - \sum \bar{x}_w^k]}{n}$ \{Projection\}
7: \hspace{2em} while $\bar{x}_w^k < 0$ do \{Any path flow element is negative\}
8: \hspace{3em} $J = \{j : \bar{x}_w^k > 0\}$ \{Indices of positive path flow elements\}
9: \hspace{3em} $\bar{x}_w^k = \bar{x}_w^k + \frac{[d_w^k - \sum \bar{x}_w^k(J)]}{|J|}$ \{Include only positive elements\}
10: \hspace{3em} $\bar{x}_w^k(J) = \bar{x}_w^k(J)$ \{Update\}
11: \hspace{3em} $\bar{x}_w^k(J) = 0$ \{Elements not in $J$ set to zero\}
12: \hspace{2em} end while
13: \hspace{1em} end for
14: \hspace{1em} end for

3.4 A Multi-modal and Multi-actor Equilibrium Model

To analyse the impact on the transportation system of different MSP strategies, the problem can be formulated as an Equilibrium Problem with Equilibrium Constraint (EPEC). An EPEC can be considered as an extension of a Stackelberg game (Stackelberg, 1952), where a Stackelberg game is characterised by the presence of a (single) leader whose strategy impacts the decisions made by a group of followers that play a non-cooperative Nash game between one another (Nash, 1951). At the upper-level, EPECs are charac-
terised by the presence of multiple leaders who participate in their own Nash game (Steffensen and Bittner, 2014). Generally, each leader in an EPEC aims to solve their own Mathematical Program with Equilibrium Constraints (MPEC) (Luo et al., 1996). Stackelberg games are naturally formulated as MPECs, where the lower-level equilibrium constraints may be formulated as a Variational Inequality (VI) or as Karush Kuhn Tucker (KKT) conditions (Luo et al., 1996). As a result, EPECs can be considered as a collection of MPECs that share variables and equilibrium constraints (Cottle and Su, 2005).

EPECs have rarely been used in transportation problems. Yang et al. (2009) introduced an EPEC that examines how competing firms maximize their profits by adjusting road capacity and toll charges, while homogeneous users are assigned to the network. They employed a heuristic method called the Synchronous Iterative Method to solve the problem. Koh and Shepherd (2010) investigated a similar problem by formulating an EPEC where firms generate revenues by imposing tolls on transportation network users. They proposed two heuristic solution algorithms: the Diagonalization Algorithm and the Sequential Linear Complementary Programming (SLCP) approach. Subsequently, Koh et al. (2013) analysed the competition between two city authorities. The objective was to maximize the social welfare of their respective residents while charging traffic through tolls. Results show that increasing elasticity, changing demand and congestion functions their model can shift from multiple equilibrium solutions to a single solution. Watling et al. (2015) extended these previous works casting the problem as a single level problem for each authority, in order to explore the scenario where each authority seeks a local optimum within their individual MPEC. Finally, Gu et al. (2019) structured this problem as a tri-level optimization model, considering the government at the upper-level, who seeks the optimal toll to maximize social welfare. Private firms try to maximize their profits based on the government’s decisions, charging tolls and investing in road capacities. Lastly, users are assigned to the network following User Equilibrium. The problem between private firms and travellers has the structure of an EPEC and is solved through a synchronous iterative method. Finally, Wang et al. (2021) studied the competition between the government and a private firm in a public-private mixed network, under elastic demand. The government’s objective function maximizes social gains on public roads, instead the firm tries to maximize profit through tolls on some controlled roads. The study uses the Diagonalization method to solve the problem. Recently, in the context of the Crowdsourced Event Parking Market Pricing Problem, Fotouhi and Miller-Hooks (2021) formulated an EPEC to study competition between parking space owners (upper-level) that want to optimize parking charges offered to (lower-level) users who can accept these prices or alternatively choose a mode of transport that doesn’t require to pay for parking. They also solved the model using a Diagonalization approach.

As stated previously, an EPEC is a problem that can be broken down into a set of connected MPECs. More precisely, in the problem under analysis, instead of tackling the entire complex problem as one unit, it can be divided into individual problems, with each MSP evaluating their optimal solution via a specific MPEC, which simplifies the overall analysis and solution process.

Following this approach, a change of a MSP’s fleet size determines a new equilibrium distribution of the lower-level. This new distribution, in turn, affects the profits of all MSPs at the upper-level. Given that each MSP aims to maximize their profit while considering the strategies of other players, equilibrium is achieved when no MSP or users have any incentive to change their strategies, as doing so would put them in a less advantageous position. Therefore following Equation (6), each MSP $j \in J$ selects their fleet size, $v^j$, to maximize their own profit, $Z^j$. The profit of each MSP depends on the path flows of the lower-level. The equilibrium path flows depend on the fleet sizes of all the MSPs, $x^*(v)$. With the lower-level equilibrium constraint in place it is possible to consider that the profit of the $j$-th MSP is a continuously differentiable function, $Z^j(v^j|x^*(v))$. Naturally, each MSP can change only their own fleet size; dependence on other MSPs fleets comes through the lower-level equilibrium condition.
Equilibrium at the upper-level is reached when no MSP can unilaterally change their fleet size and increase their profit. An upper-level equilibrium is denoted $\tilde{v}$, whose (scalar) $j$-th component is $\tilde{v}^j$. When the $j$-th component (only) of $\tilde{v}$ is set to the value $\rho$, it is possible to write $\tilde{v}^j = \rho$. The upper-level equilibrium condition can be written for $\tilde{v}$ as follows:

$$Z^j(\tilde{v}^j|x^*(\tilde{v})) - Z^j(\rho|x^*(\tilde{v}^j = \rho)) \geq 0 \quad \forall \rho \geq 0 \quad \text{for each } j \in \mathbb{J}$$

(18)

The lower-level equilibrium path flows $x^*(\tilde{v})$ are defined above (see Equation 15). Recall that uniqueness is not guaranteed for the lower-level equilibrium due to the non-separability of the cost functions; there might be multiple path flows solutions for a given vector of fleet sizes. Clearly, this reflects in also multiple solutions for the upper-level, as it will be shown in the numerical examples.

**Solution Algorithm: the Diagonalization Method.** EPEC problems are well known in literature for the difficulty of finding equilibrium solutions. Moreover, there are no specifically designed algorithms to solve EPECs (Cottle and Su, 2005). As pointed out by Watling et al. (2015) and citations therein, two classes of methods are generally used to solve EPECs: the Simultaneous Methods (SM) and the Diagonalization Methods (DM). SMs try to solve the different MPECs simultaneously, while DMs “solve a cyclic sequence of single-leader-follower games until the decision variables of all leaders reach a fixed point” (Leyffer and Munson, 2010). Specifically, a DM iteratively solves each MPEC in turn, while keeping fixed the decision variables of all other players. Common examples of DMs are the Gauss-Jacobi and Gauss-Seidel methods, initially used to solve VI problems (Harker, 1984). In the first approach, all variables are updated simultaneously using the values from the previous iteration, without updating values within the same iteration. The second approach, instead, updates each variable using the most recent values, making it more efficient in terms of convergence but less amenable to parallelization due to its sequential nature.

Generally, solving MPECs presents significant challenges, particularly due to the presence of disjoint constraints that give rise to combinatorial problems, posing difficulty for efficient solution algorithms. More likely, the lack of convexity and closedness of the feasible region could be a cause of inefficiencies in finding optimal solutions for MPECs (Luo et al., 1996). In this paper a DM with the structure of a Gauss-Seidel method is used, where for each MSP in turn an MPEC formulation is solved. Considering that the order in which the MSP-specific are tackled is arbitrary in the DM.

In transportation usually link-separable cost functions are considered at the lower-level in order to simplify the problem, and reducing the VI into a convex optimization problem. This allow to write the problem as a bi-level program that can be restructured into a single-level convex optimization problem. The benefit of this simplification is that it allows for the use of a wider range of existing optimization algorithms that are designed for convex optimization. However, in the proposed formulation, the lower-level multi-class UE is defined assuming that the different costs perceived by users are not-separable. This aspect increases drastically the complexity of the model, making the MPEC hard to be solved as an optimization problem using conventional solution algorithms.

The choice of algorithm depends on the specific characteristics of the transportation problem, such as the network structure, the type of users and decisions involved, and convexity properties. In this paper, to solve the proposed problem introduced in Equation (6), subject to constraint (15), an iterative solution algorithm is proposed: it searches for a local solution of the upper-level continuous objective function using a gradient-based method, based on the sequential quadratic programming (SQP) algorithm (Wright et al., 1999). This algorithm is an iterative optimization algorithm commonly used for solving nonlinear constrained optimization problems. Specifically, it solves a series of quadratic sub-problems that approximate the original nonlinear problem and move iteratively towards the optimal solution.
In the general iterative process, as each upper-level function evaluation occurs, a lower-level equilibrium solution is determined through the adaptive EM method described in Section 3.3. The sequence of steps for this procedure is detailed in Algorithm 1 and Algorithm 2. Algorithm 3 shows the different steps of the proposed iterative DM.

**Algorithm 3 Diagonalization Method**

1: Input Parameters and Functions \( \forall J, K \)
2: Set \( \text{MaxIter}, \varepsilon, v_0, v_{\min}, v_{\max} \) \{Initialization\}
3: Set \( i = 1, \Delta = \infty, v_i = v_0 \)
4: while \( \Delta > \varepsilon \) and \( i < \text{MaxIter} \) do \{Stopping Criteria\}
5: for \( j = 1:J \) do \{Number of MSPs\}
6: Compute \( y \in [v_{\min}^j, v_{\max}^j] \) maximizing \( Z^j(y|x^*(v_i[v^j = y])) \)
   with \( x^*(v_i[v^j = y]) \) through EM using Algorithm 1
7: Update \( v_i^j = y \) \([j\text{-th component}]\)
8: end for
9: Compute \( \Delta = \|v_i - v_{i-1}\| \)
10: Set \( i = i + 1 \)
11: end while

4 Application Examples

In this section, we aim to showcase the application opportunities and the complexity of the problem under study by implementing first the proposed approach on a first example (Figure 5). This allows us to highlight the feasibility of the approach and provide initial evidence of its practicality, laying the groundwork for its potential implementation in more complex scenarios. In order to demonstrate certain features of a MaaS system, such as the inclusion of mobility packages, which can be represented using the suggested approach, we utilise the solution algorithm in another scenario, illustrated in a second example (Figure 14a).

The algorithms presented in Section 3.3 were implemented in MATLAB R2023\(^2\). The upper-level profit maximization used the MATLAB optimization toolbox (MATLAB, 2023).

4.1 Example 1: Competition between MSPs

We consider the equilibrium emerging between two car-sharing services competing in a region. For the sake of solution interpretability, a single OD pair is considered in this first case, where in addition to car-sharing, users can choose between public transport and private car. We consider three classes of users performing the same trip chain (Figure 5a) between two locations: L1 to L2 and returning to L1. Different classes could represent users performing different activities (for instance Home-Work, Home-School and Home-Leisure) that give rise to different perception of trip costs (e.g. different values of time). Classes may also represent different perceived travel utility due to socio-economic characteristics. Users are assigned to the four modes connecting the two locations, as shown in Figure 5b.

\(^2\)The simulations are carried out using Windows 10 laptop with an Intel(R) Core (TM) i7-8650U CPU with a base frequency of 1.90GHz and a system memory of 16.0 GB.
We consider that all modes of transport share the same road infrastructure, therefore link cost are assumed non-separable. The link cost components all take the form of the conventional Bureau of Public Roads (BPR) function to include congestion effects, or are constant. We focus on the competition between the two car-sharing service providers, naming them car-sharing 1 \((j = 1)\) and car-sharing 2 \((j = 2)\). The two services mainly differ in terms of the offered package price. Car-sharing 1 charges a higher monthly package fee, whereas car-sharing 2 offers a service at a lower price, thanks to additional revenues received from e.g. advertising. Hence, car-sharing 2 is potentially a more attractive service than car-sharing 1 for the users, provided that they offer the same capacity (which depends on the deployed fleet size).

<table>
<thead>
<tr>
<th>Table 3: Algorithms Parameters (see Algorithm 1 and Algorithm 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Algorithm 3</strong></td>
</tr>
<tr>
<td>Parameters</td>
</tr>
<tr>
<td>MaxIter</td>
</tr>
<tr>
<td>(\varepsilon)</td>
</tr>
<tr>
<td>MinFleetSize</td>
</tr>
<tr>
<td>MaxFleetSize</td>
</tr>
<tr>
<td>(\varepsilon)</td>
</tr>
</tbody>
</table>

In Table 3, the parameters from the different algorithms used to evaluate the EPEC are listed. In Table 4, we introduce the parameters associated to each user class and the costs for the services provided. Finally, Table 5 lists the parameters associated to the different cost functions.
Table 4: Example 1: Cost and Demand Parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Private Car</th>
<th>Bus</th>
<th>Car-sharing 1</th>
<th>Car-sharing 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_s )</td>
<td>0.1</td>
<td>1.5</td>
<td>0.8</td>
<td>0.5</td>
</tr>
<tr>
<td>( r_s )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( c_{a,\text{access}}^{1} )</td>
<td>8</td>
<td>11</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>( c_{a,\text{egress}}^{1} )</td>
<td>7.2</td>
<td>9.9</td>
<td>8.1</td>
<td>8.1</td>
</tr>
<tr>
<td>( c_{a,\text{access}}^{2} )</td>
<td>11.7</td>
<td>15.4</td>
<td>12.6</td>
<td>12.6</td>
</tr>
<tr>
<td>( c_{a,\text{wait}}^{1} )</td>
<td>-</td>
<td>13</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>( c_{a,\text{wait}}^{2} )</td>
<td>-</td>
<td>11.7</td>
<td>8.1</td>
<td>8.1</td>
</tr>
<tr>
<td>( c_{a,\text{wait}}^{3} )</td>
<td>-</td>
<td>18.2</td>
<td>12.6</td>
<td>12.6</td>
</tr>
<tr>
<td>( c_{a,\text{main}}^{1} )</td>
<td>8.2</td>
<td>9.9</td>
<td>8.5</td>
<td>8.5</td>
</tr>
<tr>
<td>( c_{a,\text{main}}^{2} )</td>
<td>7.4</td>
<td>8.9</td>
<td>7.6</td>
<td>7.6</td>
</tr>
<tr>
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<td>13.8</td>
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<td>11.9</td>
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<tr>
<td>( c_{a,\text{fuel}} )</td>
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<td></td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>( c_{a,h} )</td>
<td>-</td>
<td></td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>( c_{a,\text{km}} )</td>
<td>-</td>
<td></td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>( l_a )</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>( c_{a,\text{park}}^{1} )</td>
<td>11</td>
<td></td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( c_{a,\text{park}}^{2} )</td>
<td>9.9</td>
<td></td>
<td>-</td>
<td>-</td>
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<tr>
<td>( c_{a,\text{park}}^{3} )</td>
<td>15.4</td>
<td></td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( v_a )</td>
<td>300</td>
<td>400</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( d_1^{1} )</td>
<td></td>
<td></td>
<td>300</td>
<td></td>
</tr>
<tr>
<td>( d_2^{1} )</td>
<td></td>
<td></td>
<td>200</td>
<td></td>
</tr>
<tr>
<td>( d_3^{1} )</td>
<td></td>
<td></td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

Table 5: Example 1: Parameters of the cost functions

<table>
<thead>
<tr>
<th>Function</th>
<th>Private Car</th>
<th>Bus</th>
<th>Car-sharing 1</th>
<th>Car-sharing 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_{a,\text{access}}(f_a,v_a) )</td>
<td>( t_0 )</td>
<td>( \alpha )</td>
<td>( f )</td>
<td>( C )</td>
</tr>
<tr>
<td>( t_{a,\text{wait}}(f_a,v_a) )</td>
<td>-</td>
<td>0.1</td>
<td>-</td>
<td>0.1</td>
</tr>
<tr>
<td>( t_{a,\text{main}}(f) )</td>
<td>0.2</td>
<td>2</td>
<td>( f )</td>
<td>600</td>
</tr>
<tr>
<td>( t_{a,\text{park}}(f_a,v_a) )</td>
<td>0.1</td>
<td>1</td>
<td>( f_a )</td>
<td>300</td>
</tr>
<tr>
<td>( t_{a,\text{egress}}(f_a,v_a) )</td>
<td>0.08</td>
<td>-</td>
<td>-</td>
<td>0.1</td>
</tr>
<tr>
<td>( c_{\text{lease}}(v) )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

BPR functional form is: \( t = t_0 \left( 1 + \alpha \left( \frac{f}{C} \right)^\beta \right) \)

Figure 6 shows the profit surfaces of the two suppliers, obtained by computing the lower-level equilibrium flows at each combination of fleet sizes for car-sharing 1 and car-sharing 2. The dots represent the solutions of the diagonalization method for solving the EPEC starting from different initial points (these equilibrium points are also reported in Table 6). The computational time needed to reach an equilibrium solution varies based on the starting point. The closer the starting point is to the solution, the less time is required to find it. For instance, for the tested initial points, the computational time ranged from 135 seconds to 1143 seconds. These plots and the emergence of multiple equilibria with different computing times show the non-linearity and non-convexity of the profit function surfaces, in particular revealing clear
regions where the marginal profit changes at different rates with a change of fleet size(s), and regions where one or both services are not profitable (green areas).

![Figure 6: Profit variation with equilibrium points](image)

(a) Car-sharing 1  
(b) Car-sharing 2

In Table 6 the best compromise solution for both suppliers is highlighted in red, where equal profit is achieved when offering approximately the same fleet size. Moreover, due to the symmetry of the network, the distribution of vehicles in each link is the same at equilibrium. It is worth noting that all possible solution points are located within the two primary areas characterized by higher profitability for both suppliers, which can be clearly identified in Figure 6 (all black dots).

Figure 6 provides an illustration of how the order in which the MSPs’ objective functions are evaluated in the Diagonalization approach determines the resulting solution points. Specifically, the green dots illustrate the solutions where, starting from the same initial point (0,0), car-sharing 1 is optimized first (left graph), and where car-sharing 2 is optimized first (right graph), respectively. Hence, applying the diagonalization approach but varying the order in which the suppliers optimise their fleet to seek maximum profit, we observe distinct solution points at equilibrium. This finding emphasises the critical role played by the order chosen in the solution algorithm, as it significantly impacts the final solution.

Apart from an interpretation of these results from a computational standpoint, this solution non-uniqueness may have a practical implication; the solution found by prioritising first car-sharing 1 could be seen as a potential equilibrium state emerging when car-sharing 2 enters the market only in a later stage than car-sharing 1, i.e. a new service entering the market that competes with an existing service. Hence, this new service provider may be able to reach the maximum achievable level of profitability only by heavily lowering the fleet with respect to the equilibrium achieved, but this strategy may not be observable when being on the achieved equilibrium state as any smaller reduction or increase of the fleet would result in a potential profit loss.
When examining the total profit surface (Figure 7a), obtained by simply summing up the two individual profit surfaces (shown on Figure 7b), we observe that the equilibrium points deviate from the maximum profits that both car-sharing operators could attain if they pursued an entirely cooperative approach.
This observation reveals that the pursuit of individual profit optimization is likely to result in a sub-optimal outcome compared to a fully collaborative strategy. This naturally prompts us to evaluate the Price of Anarchy (PoA) (Koutsoupias and Papadimitriou, 1999; Papadimitriou, 2001), which is a common measure of the inefficiency of a system in which actors are playing in a self-interested manner. Namely, PoA is the ratio between the worst-case Nash Equilibrium, where there is no coordination, and the optimal system performance that would be achieved if the players were compelled to coordinate their actions (Christodoulou, 2008). The PoA in this case corresponds to the worst possible profit ratio: summed profits of MSPs at the worst-case Nash equilibrium (blue point in Table 6) divided by the maximum attainable total profit. Note that we cannot guarantee that we have computed all possible equilibria. A smaller numerator for Eq. 19 might exist, and hence we cannot evaluate the PoA (the infimum). Nevertheless, the computed points give an upper bound on the PoA:

\[
PoA \leq \frac{\min \sum_{NE} Z^j}{\max \sum_{oc} Z^j} = \frac{177.1}{229.7} = 0.77
\]

implying that when the system exhibits non-cooperative behaviour, it may diminish the maximum potential profitability by 23% (or more) compared to what could be achieved through cooperation.

After analysing the equilibrium solutions computed above, and examining the profit surfaces, a notable observation emerges: there exist two distinct regions of equilibria that appear to constitute continuous sets of solutions aligned along two primary lines. A least squares fit to the multiple equilibrium points computed reveals that these lines have equations:

\[
v^2 = -0.9517v^1 + 159.1468
\]

for the upper line, and:

\[
v^2 = -1.0309v^1 + 80.7030
\]
for the lower line; where \( v^1 \) is the fleet size of car-sharing 1, and \( v^2 \) is the fleet size of car-sharing 2.

From a visual inspection of the surfaces displayed in Figure 6 it may appear that equilibrium solutions occur all along these lines from \( v^1 \in [0, 69] \) for the lower line and approximately \( v^1 \in [0, 104] \) for the upper line. However, numerical investigations suggest this is not the case. Each linear set of solutions has boundary points, shown by the line-end dots in Figure 8. There appear to be a continuum of equilibrium points within each marked segment. However, points on the extrapolation of these line segments, but outside the marked end points, do not meet the equilibrium criteria and when used as initial points for the algorithm, the nearest boundary point is returned as the equilibrium solution, as shown by the yellow arrow in Figure 8. This requires further investigation, to understand how continuous or separated equilibria arise in different scenarios.

![Figure 8: Profit variation with equilibrium points](image)

Generally, this example provided insights into the complexity of competitive transportation networks with interacting decision-making processes of the MSPs. Furthermore, the impact of different user classes on the model was explored, demonstrating how changes in demand between these classes influenced profit surfaces due to shifts in path flows based on distinct cost perceptions.

Focusing on the users, we can investigate the impact of capacity changes on the total travel cost (TTC) of the network, as shown in Figure 9. It is straightforward to observe that increasing the fleet sizes, i.e. increasing the service capacity, leads to a reduced total travel cost, since the services quality improves in terms of vehicle availability in space and time. Furthermore, Figure 9 reprints the position of the equilibrium solutions relative to the network travel cost. Moreover, on the last column of Table 6 we show for each equilibrium point the corresponding value of the TTC. We can see how this value tends to remain constant for all the equilibrium points positioned in the same diagonal, with higher values when there is higher profit for the suppliers. This means that the points corresponding to the maximum possible profit for the suppliers does not correspond to the most convenient solutions for the customers. Although the system cost shows only slight variations in this analysis, it can still provide valuable insights for government entities or public authorities seeking to assess the achievement of societal targets. By considering the relationship between equilibrium solutions and network travel cost, policymakers can gain a better understanding of the overall
impact and effectiveness of their transportation policies in meeting societal objectives, e.g. seeking to push more travellers to shared mobility options.

We further compare the profit for each car-sharing service, in turn, operating as a monopoly within the network. This is shown in Figure 10 by the two continuous lines, which have been computed assuming that the other service has no vehicles available.

Several equilibrium solutions are also shown for sake of comparison, highlighting the potential trade-offs between the monopolistic scenarios and market competition. All feasible equilibrium solutions in the competitive scenario result in reduced profitability for both MSPs. Moreover, we can observe that a similar profitability could be potentially attained by the services with around 60 vehicles, whereas car-sharing 2 has also another local point of maximum when deploying around 180 vehicles.

In order to fully understand these discontinuities in the derivatives of the profit functions, we further analyse the mode shares for all four mode alternatives (car-sharing 1, car-sharing 2, car and bus). Figure 11a shows the path flow variation of the two car-sharing services, with car and bus system in Figure 11b.

We can see, straightforwardly, how the maximum path flow on the two services is reached when the MSPs are offering the maximum fleet size. Additionally, these figures clearly illustrate how the discontinuity observed in the profit surfaces is reflected in the variations of path flow. Specially, the flow variation on car-sharing 1 presents the same behaviour as the profit of Figure 6 starting from a fleet size of around 100 vehicles. However, we can see on Figure 11b that below the same value there is an increase in path flow on both private car and bus mode alternatives. It seems also evident from the flow graphs that the second peak for the car-sharing 2 system acting as monopoly is due to attracting the private car demand, but above a certain fleet size the extra demand attracted is not resulting in sufficient revenues that would justify the additional costs.

It also quite interesting to observe that the flow variation on car-sharing 2, unlike the other services, has a smoother trend, clearly due to the different service cost rate. Moreover, in both car-sharing services there is a clear monotonic increase in the demand that corresponds to the first and highest peaks of MSP’s profit variations. Finally, it is clear from Figure 11b that the maximum flow on private car and bus corresponds to
the minimum fleet sizes of the car-sharing services, as expected. On the contrary, when the capacity of the sharing systems is at its maximum, the usage of private car and bus becomes constant, i.e. increasing the fleet sizes will only result in additional costs but no extra revenue will be generated.

![Flow variations for all four modes in Example 1](image)

Figure 11: Flow variations for all four modes in Example 1

Our methodology makes it also possible to analyse different scenarios, for instance in terms of supply and demand characteristics.

We first investigate how car-sharing 1 can increase its own profit in the network. For this reason, we decrease the fixed cost of the subscription from $c_s = 0.8$ to $c_s = 0.7$ and the service cost per km from $c_{a,h} = 0.6$ to $c_{a,h} = 0.4$. The results regarding the profit variation with fleet sizes are shown in Figure 12. We can see that generally the profit has increased for both suppliers, especially for car-sharing 1. Interestingly, regardless of the initial starting point, the model consistently converged to the same equilibrium point. Notably, by merely changing some parameters specification for one of the players, the equilibrium solution has completely different characteristics from the previous scenario, that appears to be a unique solution for this problem. In Table 7 we explicitly display the results for an arbitrary starting point. The PoA in this case has a higher value, meaning that the Nash equilibrium solution has a value that is closer to the maximum profit of the system.
Additionally, we analyze the impact of the different user classes on the model, by keeping the same total demand and changing the value of the demand in between class 1 and class 3: \(d_1^1 = 100\) and \(d_1^3 = 300\). In Figure 13 the surfaces of the two profit variations are even smoother than the previous scenarios, suggesting that the composition of the different classes strongly influences the distribution of flow and the
ultimate profit for suppliers. Looking at both Figure 12 and Figure 13 it becomes evident to observe that the equilibrium point reached in both cases is the (only) point in the explored range of fleet sizes where the profit curves are orthogonal to each other, and concurrently the partial derivative of each respective profit function is zero.

In Table 8 we explicitly display the results for an arbitrary starting point, where we can see an important increase in the TTC compared to the previous cases.

<table>
<thead>
<tr>
<th>Initial Fleet Sizes</th>
<th>EPEC Solution</th>
<th>Profit</th>
<th>PoA</th>
<th>TTC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fleet Size 1</td>
<td>Fleet Size 2</td>
<td>Fleet Size 1</td>
<td>Fleet Size 2</td>
<td>MSP1</td>
</tr>
</tbody>
</table>

4.2 Example 2: Mobility package introduction

In this second example we study the impact of the introduction of a mobility package that integrates the offer from two MSPs. This analysis is performed on a slightly more complex scenario where we include different characteristics described in the methodology with respect to Example 1, resulting in a more complex supernetwork.

We considered the transport network illustrated in Figure 14a, in which we included two ODs. This network is composed by 39 nodes and 52 links. Three classes of users are again taken into account. This time one class represents the demand on the first OD pair, which may represent users chaining two trips in their daily routine, and two classes are on the second OD pair. Four modes of transport are again available: private car, bus and two car-sharing services. Car-sharing 1 service is operating only on OD2, while car-sharing 2 is available everywhere. In this particular scenario, we explored the integration of a mobility package wherein the bus provider and car-sharing 1 supplier collaborate to offer an integrated solution. This mobility package is represented by the sub-graph inside the red-dashed box in Figure 14b. The transport network is designed to enable users to subscribe to a monthly package that combines bus and car-sharing 1 services, or alternatively, they can choose the bus-only option at a reduced price compared to the pay-as-you-go alternative. We considered that this package is subsidized by a local authority, and in order to reduce car congestion, users cannot use only car-sharing 1 when subscribing to the package, but they have to at least make a trip with the bus service. Hence, in this example we considered the competition strategies of the two car-sharing suppliers when one of them starts a collaboration with a third service. The algorithm parameters used are the same listed in Table 3. In Table 9, we introduce the parameters associated to user classes and costs for the services provided. Finally, Table 10 lists the parameters associated to the different cost functions. We considered that the bus service has a dedicated lane, therefore separable cost functions with respect to the other modes of transport that, instead, are considered to influence each other in congestion.
Figure 14: Example 2

(a) Daily trip chains and available modes of transport

(b) Supernetwork expansion
Table 9: Example 2: Cost and Demand Parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Private Car</th>
<th>Bus PAYG</th>
<th>Bus Package</th>
<th>Bus + Car-sharing 1 Package</th>
<th>Car-sharing 1</th>
<th>Car-sharing 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_s )</td>
<td>0.1</td>
<td>1.1</td>
<td>0.7</td>
<td>0.9</td>
<td>1.2</td>
<td>1.3</td>
</tr>
<tr>
<td>( r_s )</td>
<td>-</td>
<td>-</td>
<td>0.5</td>
<td>0.5-1</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( c_{a,\text{access}}/c_{a,\text{egress}}^1 )</td>
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<td>11</td>
<td>11</td>
<td>11-9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>( c_{a,\text{access}}/c_{a,\text{egress}}^2 )</td>
<td>8</td>
<td>11</td>
<td>11</td>
<td>11-9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>( c_{a,\text{access}}/c_{a,\text{egress}}^3 )</td>
<td>10.4</td>
<td>14.3</td>
<td>14.3</td>
<td>14.3-11.7</td>
<td>11.7</td>
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</tr>
<tr>
<td>( c_{a,\text{wait}} )</td>
<td>-</td>
<td>11</td>
<td>11</td>
<td>11-9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>( c_{a,\text{wait}} )</td>
<td>-</td>
<td>11</td>
<td>11</td>
<td>11-9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>( c_{a,\text{main}} )</td>
<td>8.2</td>
<td>9.5</td>
<td>9.5</td>
<td>9.5-8.5</td>
<td>8.5</td>
<td>8.5</td>
</tr>
<tr>
<td>( c_{a,\text{main}} )</td>
<td>8.2</td>
<td>9.5</td>
<td>9.5</td>
<td>9.5-8.5</td>
<td>8.5</td>
<td>8.5</td>
</tr>
<tr>
<td>( c_{a,\text{main}} )</td>
<td>10.7</td>
<td>12.3</td>
<td>12.3</td>
<td>12.3-11</td>
<td>11</td>
<td>11</td>
</tr>
<tr>
<td>( c_{a,fuel} )</td>
<td>0.37</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>( c_{a,h} )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.4</td>
<td>0.9</td>
<td>0.7</td>
</tr>
<tr>
<td>( c_{a,km} )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.4</td>
<td>0.5</td>
<td>0.4</td>
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<tr>
<td>( l_a )</td>
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<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>( c_{a,\text{park}} )</td>
<td>11</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( c_{a,\text{park}} )</td>
<td>11</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( c_{a,\text{park}} )</td>
<td>14.3</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
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<tr>
<td>( d_1 )</td>
<td>300</td>
<td>800</td>
<td>800</td>
<td>800</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( d_2 )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
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<tr>
<td>( d_3 )</td>
<td>200</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 10: Example 2: Parameters of the cost functions

<table>
<thead>
<tr>
<th>Function</th>
<th>Private Car</th>
<th>Bus</th>
<th>Car-sharing 1</th>
<th>Car-sharing 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_{a,\text{access}}(f_a, v_a) )</td>
<td>( t_0 )</td>
<td>( \alpha )</td>
<td>( f )</td>
<td>( C )</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>0.1</td>
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<td>0.1</td>
</tr>
<tr>
<td>( t_{a,\text{wait}}(f_a, v_a) )</td>
<td>-</td>
<td>0.1</td>
<td>( f_a )</td>
<td>800</td>
</tr>
<tr>
<td>( t_{a,\text{main}}(f) )</td>
<td>0.2</td>
<td>2</td>
<td>( f )</td>
<td>600</td>
</tr>
<tr>
<td>( t_{a,\text{park}}(f_a, v_a) )</td>
<td>0.1</td>
<td>1</td>
<td>( f_a )</td>
<td>300</td>
</tr>
<tr>
<td>( t_{a,\text{egress}}(f_a, v_a) )</td>
<td>0.08</td>
<td>-</td>
<td>0.1</td>
<td>-</td>
</tr>
<tr>
<td>( c_{\text{lease}}(v) )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

We apply the proposed algorithm to the provided example, obtaining a unique solution from different starting points. To assess the impact of introducing the mobility package within the network, we compared the results of the proposed example with a scenario where no mobility package was introduced between the bus and car-sharing 1. In particular, we considered a full price for car-sharing 1 in every link, while the bus service offered two options: pay-as-you-go \( (c_s = 1.1) \) or a monthly package \( (c_s = 0.9) \). In the scenario without the package, car-sharing 1 has no vehicles on the network because there are not enough users traveling with this mode of transport at equilibrium. However, after introducing the package, the provider has Link 3 and Link 4 inactive, and instead, in Link 1 and Link 2, 93 and 35 vehicles respectively. On the scenario without package, car-sharing 2 has 47 vehicles on Link 7 and 8, and after the introduction of the mobility package this value decreases to 15.

Given the intricate nature of representing profit variations in a two-dimensional plot, which is influenced by multiple variables for each supplier, we first set the number of vehicles in the unused links to zero to show the results. Then, for each supplier, we varied the number of vehicles on the remaining links while

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*ISTTT25*  
*Bandiera et al.– Mobility Service Providers’ Strategies*  
30
keeping fixed the competitor’s vehicle distribution on the different links. The results are illustrated in Figure 15. In the figure a clear transformation of the network equilibrium is observed due to the introduction of the mobility package. In the absence of a package, car-sharing 1 fails to sustain profitability and exits the market, while car-sharing 2 manages to maintain a viable fleet and generate positive profits. However, with the introduction of the mobility package, car-sharing 1 becomes more appealing due to its collaboration with the bus service. Conversely, car-sharing 2 experiences a significant decline in profitability.

On the user’s side, we observe a small reduction of the TTC (from 20000 to 19820), when introducing the mobility package. Furthermore, Table 11 presents a comparison of flow variations at equilibrium between the two cases, with and without the MaaS package. As previously mentioned, the attractiveness of car-sharing 2 diminishes, and there is a slight decrease in private car usage. Notably, all options involving the presence of the bus service demonstrate an increase in flow, with a rising preference for car-sharing 1.
It is interesting to observe how, through this analysis, local authorities have the opportunity to apply into different strategies that can effectively increase public transport usage in comparison to private car. The results also demonstrate that with appropriate incentives, individuals may be more inclined to subscribe to MaaS. In the long run, this could have a positive impact on reducing car ownership and increasing loyalty towards sustainable modes of transportation.

5 Conclusion

This research contributes to the understanding of the interactions between MSPs and users in complex transportation system such as MaaS systems. The problem has been formulated as an EPEC. At the upper-level, a profit maximization formulation has been proposed to model the business strategies of MSPs, aiming to optimize their respective profits. At the lower-level, a multi-class and multi-modal network equilibrium has been formulated as a VI problem, representing the daily trip choices made by users. To tackle the complexity of the resulting mathematical problems, an iterative solution approach based on the Diagonalization method has been devised. Additionally, an adaptive EM algorithm has been incorporated to address the lower-level network equilibrium problem.

The proposed solution algorithm has been applied to simplified example scenarios. Specifically, we proposed a multi-modal network in which private car, bus and two car-sharing services were available transport options. Three different classes of users performing the same trip chain were assigned to the cited network. The study focused on the competition between the two car-sharing service providers. The profitability of the car-sharing providers was analyzed by computing the lower-level equilibrium flows at various fleet sizes. The profit surfaces of the two suppliers demonstrated the non-linearity and non-convexity of the profit function, revealing regions where the marginal profit changed at different rates with changes in fleet size. Multiple distinct EPEC solutions were obtained, highlighting the non-uniqueness of the equilibrium points. Moreover, the analysis of the profit surfaces and equilibrium points emphasized the impact of the prioritization of car-sharing services within the diagonalization process. Generally, this example demonstrated the complexity and multiple equilibria that can arise in a competitive transportation network, shedding light
on the considerations and decision-making processes for MSPs. Finally, we showed the impact of the user classes on the model, changing the demand between two classes had an impact on both profit surfaces, due to the new distribution of path flows based on the different cost perception.

Additionally, we showed how the proposed methodology can be applied to a slightly more complex network in which cooperation through a mobility package is introduced, and competition has been evaluated. The results have implications for understanding market dynamics and developing strategies in the transportation market. Clearly many more decision variables could be introduced and examined under the framework established here, giving rich opportunities for future research.

The proposed methodology could be for instance applied to analyse more complex scenarios, concerning new MSPs entering in the market, adopting different pricing policies or more complex package options including several MSPs, and heterogeneous MSPs competing at the upper level. This work could be extended further by incorporating calibrated functional parameters and designing optimal pricing schemes. Currently, these schemes are treated as exogenous in the model, as prices haven’t been considered as decision variables. Introducing service prices as additional decision variables could provide valuable insights. This would allow a better understanding of how to reduce and minimize the value of PoA but also offers the opportunity to enhance service profitability, beyond just varying fleet sizes. Finally, the aim of future developments is to expand its application to bigger networks considering the competition and cooperation between multiple suppliers at the upper level with multiple user classes at the lower level. Moreover, it would be interesting to include another level in which we study the impact of the Government on MSPs’ market strategies. Through this approach it will be possible to study different dynamics that occur in the transportation network due to the presence of heterogeneous actors with diverse purposes.

Acknowledgement

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A Lower-level Numerical Testing

In this section, the proposed solution algorithm for the UE is applied to the network represented in top Figure 16 with the purpose analysing the characteristics of the different functions and their influence on the equilibrium solutions. In the proposed network a single class performs a daily tour from location $L_1$ to location $L_2$, to finally return to the first location $L_1$.

![Daily Trip Chain](image)

Three modes of transport are included: PC, bus, and a one-way car-sharing service. In the presented example it is considered that PC and the car-sharing service have non-separable cost functions due to the fact that are sharing the same infrastructure. The bus service, instead, is considered to have dedicated lanes without experiencing congestion from other modes of transportation.

Following the formulations described in Section 3.3, the parameters taken into account are listed in Table 12. In this example, the link costs functions are considered constant, or they take the form of the conventional Bureau of Public Roads (BPR) function:

$$ t = t_0 \left( 1 + \alpha \left( \frac{f}{C} \right)^\beta \right) $$

(22)

where the coefficients $\alpha$ and $\beta$ define the shape of the function, $t_0$ represents the travel time in free-flow conditions. Finally the travel time will increase based on the ratio between flow ($f$) and capacity ($C$) (Maerivoet and De Moor, 2000). Table 13 shows the different functional parameters.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>Iter.</th>
<th>Gap ($\times 10^{-9}$)</th>
<th>$\lambda$</th>
<th>Iter.</th>
<th>Gap ($\times 10^{-9}$)</th>
<th>$\mu$</th>
<th>Iter.</th>
<th>Gap ($\times 10^{-9}$)</th>
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<td>0.01</td>
<td>2355</td>
<td>9.96</td>
<td>0.01</td>
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<td>0.5</td>
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<td>0.05</td>
<td>467</td>
<td>9.85</td>
<td>0.05</td>
<td>142</td>
<td>8.97</td>
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<td>0.1</td>
<td>233</td>
<td>9.80</td>
<td>0.1</td>
<td>134</td>
<td>8.59</td>
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<td>0.3</td>
<td>107</td>
<td>8.08</td>
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<td>95</td>
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<td>103</td>
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<tr>
<td>30</td>
<td>95</td>
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<td>9.91</td>
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<td>89</td>
<td>8.41</td>
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Table 14: Parameters tested
Table 12: Parameters Scenario 1: Lower-level

<table>
<thead>
<tr>
<th>Parameters</th>
<th>PC</th>
<th>Bus</th>
<th>Car-sharing</th>
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<tbody>
<tr>
<td>$c_s$</td>
<td>0.1</td>
<td>1.1</td>
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</tr>
<tr>
<td>$r_s$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$c_{a,\text{access}}/c_{a,\text{egress}}$</td>
<td>8</td>
<td>11</td>
<td>9</td>
</tr>
<tr>
<td>$c_{a,\text{wait}}$</td>
<td>-</td>
<td>13</td>
<td>9</td>
</tr>
<tr>
<td>$c_{a,\text{main}}$</td>
<td>8.2</td>
<td>9.5</td>
<td>8.5</td>
</tr>
<tr>
<td>$c_{a,\text{fuel}}$</td>
<td>0.37</td>
<td>0.2</td>
<td>-</td>
</tr>
<tr>
<td>$c_{a,h}$</td>
<td>-</td>
<td>-</td>
<td>0.3</td>
</tr>
<tr>
<td>$c_{a,km}$</td>
<td>-</td>
<td>-</td>
<td>0.3</td>
</tr>
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<td>$l_a$</td>
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<td>10</td>
<td>10</td>
</tr>
<tr>
<td>$c_{a,\text{park}}$</td>
<td>11</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$v_a$</td>
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<td>300</td>
<td>50</td>
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</tbody>
</table>

Table 13: Functional Parameters Scenario 1: Lower-level

<table>
<thead>
<tr>
<th>Function</th>
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<th>Bus</th>
<th>Car-sharing</th>
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</thead>
<tbody>
<tr>
<td>$t_{a,\text{access}}(f_a, v_a)$</td>
<td>$t_0$</td>
<td>$\alpha$</td>
<td>$f$</td>
</tr>
<tr>
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<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$t_{a,\text{wait}}(f_a, v_a)$</td>
<td>-</td>
<td>0.4</td>
<td>0.3</td>
</tr>
<tr>
<td>$t_{a,\text{main}}(f)$</td>
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<td>2</td>
<td>$f$</td>
</tr>
<tr>
<td>$t_{a,\text{park}}(f_a, v_a)$</td>
<td>0.1</td>
<td>1</td>
<td>$f_a$</td>
</tr>
<tr>
<td>$t_{a,\text{egress}}(f_a, v_a)$</td>
<td>0.08</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 14 shows how the initial choice of $\overline{\theta}$, $\lambda$, and $\mu$ impacts the number of iterations required for convergence. It is evident that for small values of $\overline{\theta}$ the number of iterations is extremely high with relatively high values of the resulting relative gap. The corresponding equilibrium distribution of the different modes of transport and the iterations necessary to achieve the same modal distribution are shown in Figure 17. In red, PC takes almost half of the demand, where bus (blue) and car-sharing (green) have an even distribution between the two modes.

The values of parameters $\mu$ and $\lambda$ can range from 0 to 1. In Table 14 it is possible to see how these values impact the number of iterations and the relative gap. It’s noticeable that when the values are small, the number of iterations and the gap value tends to increase, especially for $\lambda$. In both situations, it appears that the optimal values typically fall between 0.3 and 0.5. This leads to fewer interactions and a smaller gap. After this analysis in Table 15 are listed the algorithm parameters used for the next results.

Table 15: Algorithms Parameters (from Algorithm 1)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<td>MaxIter</td>
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<tr>
<td>$\mu$</td>
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</tr>
<tr>
<td>$\lambda$</td>
<td>0.5</td>
</tr>
<tr>
<td>$\overline{\theta}$</td>
<td>30</td>
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<tr>
<td>$\varepsilon$</td>
<td>1e-8</td>
</tr>
</tbody>
</table>
Figure 17: Impact of $\overline{\theta}$ variation on equilibrium distribution through iterations

Subsequently, in Figure 18, it is shown the difference between the proposed case with non-separability between PC and car-sharing (Figure 18b), and a possible case in which all the modes of transport are considered to not have interaction between each other (Figure 18a). In the separable case, the flow on PC and car-sharing has higher values than the bus. This flow distribution arises because congestion relies solely on the flow within each mode of transportation. In general, the distribution of path flows appears comparable across iterations in both scenarios. Nonetheless, when non-separable congestion effects are introduced, the bus option begins to emerge as a more favorable solution, attracting a greater share of the flow. From this analysis, it is clear that neglecting the influence that different modes of transport have on each other could lead to an incorrect estimation of the flow, and possibly wrong estimations of fleet sizes on suppliers’ side.